WISC is independent from ZF

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WISC (which stands for **W**eakly **I**nitial **S**et of **C**overs) is the following statement:

> For every set X there is a set of surjections $\{p_i\colon Y_i \to X\}$ onto X such that for every surjection $q\colon Z \to X$ there is an $i \in I$ and a function $f\colon Y_i \to Z$ such that $p_i = q \circ f$.

WISC is implied by the axiom of choice and the existence of enough projectives. For more information on **WISC**, see [1] (where it is called **AMC**) and the nLab.

The aim of this note is to prove that **WISC** is independent from **ZF**. The argument is inspired by the work of Andreas Blass [2].

Theorem 0.1 **ZF** *is consistent with the statement that every limit ordinal has cofinality ω.*

Proof. By a celebrated result of Gitik [3] **ZF** is consistent with the statement that all uncountable cardinals are singular, or, in other words, that ω is the only regular cardinal. Because the cofinality of a limit ordinal is a regular cardinal, **ZF** is consistent with the statement that ω is the cofinality of every limit ordinal. $\square$

Theorem 0.2 **WISC** *is not provable in* **ZF**.

Proof. Suppose there is a family $(p_i\colon B_i \to \mathbb{N} \mid i \in I)$ of surjections onto $\mathbb{N}$ such that every surjection onto $\mathbb{N}$ is refined by one in this family. We will show that this is incompatible with the statement that every limit ordinal has cofinality ω.
Put $A = I + \{0, 1\}$, $B_0 = \emptyset$, $B_1 = \{0\}$ and let

$$f\colon \sum_{a \in A} B_a \to A$$

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be the obvious projection. We will write $W = W(f)$ for the W-type associated to f.

By transfinite induction on W, we can now define a map $m\colon W \to Ord$ as follows:

$$
\begin{aligned}
m(\sup_i(t)) &= \sup\{m(tb)\colon b \in B_i\}, \\
m(\sup_0(t)) &= 0, \\
m(\sup_1(t)) &= t(0) + 1.
\end{aligned}
$$

Observe that:

1. 0 lies in the image on m.

2. The image of m is closed under taking successor ordinals.

3. The image of m is closed under countable suprema: for if $\beta = \sup(\alpha_n)$ and $\alpha_n \in m(W)$ for every $n \in \mathbb{N}$, then

$$(\forall n \in \mathbb{N})\,(\exists w \in W)\,m(w) = \alpha_n.$$

   Weak initiality of $(f_i\colon B_i \to \mathbb{N} \,|\, i \in I)$ now implies that there is an $i \in I$ and a function $t\colon B_i \to W$ such that

$$(\forall b \in B_i)\,m(tb) = \alpha_{p_i(b)}.$$

   Hence $\beta = m(\sup_i(t))$.

So if every limit ordinal had cofinality ω, the map m would be surjective. Contradiction. $\qquad\square$

Remark 0.3 By refining this argument one can show that **WISC** implies (in **ZF**) that there are class many regular cardinals and hence (by Proposition 2 on page 151 of [2]) that every variety has free algebras.

References

[1] B. van den Berg and I. Moerdijk. The axiom of multiple choice and models for constructive set theory. arXiv:1204.4045, 2012.

[2] A.R. Blass. Words, free algebras, and coequalizers. *Fund. Math.*, 117(2):117–160, 1983.

[3] M. Gitik. All uncountable cardinals can be singular. *Israel J. Math.*, 35(1-2):61–88, 1980.